# Lecture

## Discrete-Time Signals and Systems

# 1. Discrete-Time Signals and Systems

- signal classification -> signals to be applied in digital filter theory within our course,
- some elementary discrete-time signals,
- discrete-time systems: definition, basic properties review, discrete-time system classification, input-output model of discrete-time systems -> system to be applied in digital filter theory within our course,
- Linear discrete-time time-invariant system description in time-, frequency- and transform-domain.

# 1.1. Basic Definitions

## 1.1.1. Discrete and Digital Signals

### 1.1.1.1. Basic Definitions

Signals may be classified into four categories depending on the characteristics of **the time-variable** and **values** they can take:

- continuous-time signals (analogue signals),
- discrete-time signals,
- continuous-valued signals,
- discrete-valued signals.

## Continuous-time (analogue) signals:

Time: defined for every value of time $t \in R$,

Descriptions: functions of a continuous variable $t$: $f(t)$,

Notes: they take on values in the continuous interval $f(t) \in (-a, b) \;\; for \; a, b \to \infty$ .

Note: $f(t) \in C$

$$f(t) = \sigma + j\omega$$

$$\sigma \in (-a, b) \, and \, \omega \in (-a, b)$$

$$a, b \to \infty$$

## Discrete-time signals:

Time: defined **only** at discrete values of time: $t = nT,$

Descriptions: sequences of real or complex numbers $f(nT) = f(n),$

Note A.: they take on values in the continuous interval $f(n) \in (-a, b) \;\; for \; a, b \to \infty,$

Note B.: sampling of analogue signals:

- sampling interval, period: $T$ ,
- sampling rate: *number of samples per second*,
- sampling frequency (Hz): $f_S = 1/T$ .

## Continuous-valued signals:

Time: they are defined **for every value of time** or only at discrete values of time,

Value: they can take on **all possible values** on finite or infinite range,

Descriptions: functions of a continuous variable or sequences of numbers.

## Discrete-valued signals:

Time: they are defined for **every value of time** or only at discrete values of time,

Value: they can take on values from **a finite set** of possible values,

Descriptions: functions of a continuous variable or sequences of numbers.

# Digital filter theory:

## Discrete-time signals:

Definition and descriptions: defined only at **discrete** values of time and they can take **all** possible values on finite or infinite range (**sequences** of real or complex numbers: $f(n)$ ),

Note: sampling process, constant sampling period.

## Digital signals:

Definition and descriptions: **discrete-time and discrete-valued signals** (i.e. discrete -time signals taking on values from a finite set of possible values),

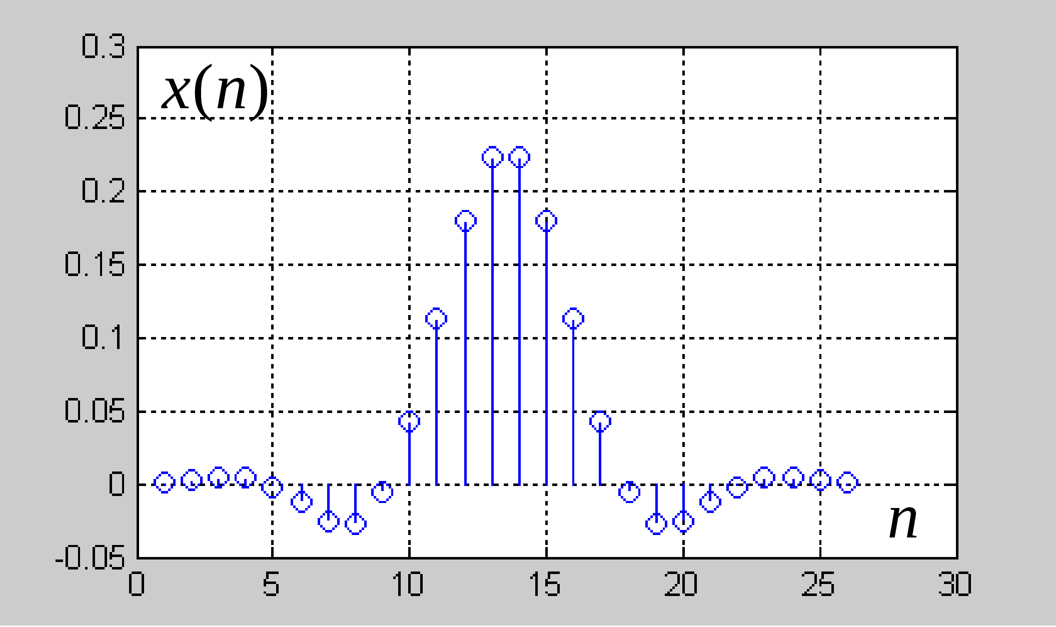
Note: sampling, quatizing and coding process i.e. process of analogue-to-digital conversion.

# 1.1.1.2. Discrete-Time Signal Representations

## A. Functional representation:

$$x(n)=\begin{cases}1 & for \quad n=1,3\\ 6 & for \quad n=0,7\\ 0 & elsewhere\end{cases}\qquad y(n)=\begin{cases}0 & for \quad n<0\\ 0{,}6^{n} & for \quad n=0,1,\ldots,102\\ 1 & \quad n>102\end{cases}$$

## B. Graphical representation



## C. Tabular representation:

| $n$ | … | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|---|
| $x(n)$ | … | 0.12 | 2.01 | 1.78 | 5.23 | 0.12 |

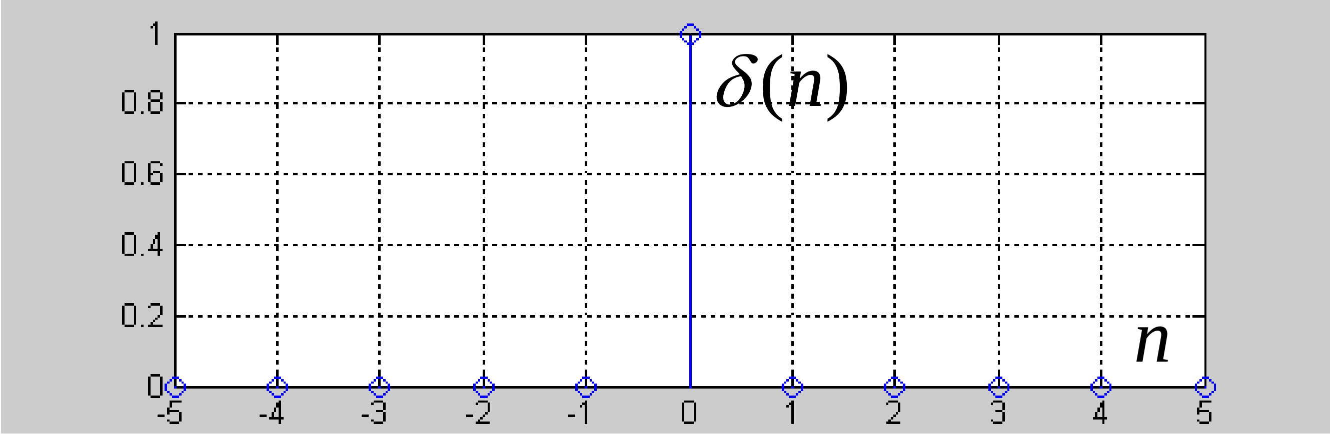
## D. Sequence representation:

$$x(n) = \{\ldots \quad 0.12 \quad 2.01 \quad 1.78 \quad 5.23 \quad 0.12 \quad \ldots\}$$

## 1.1.1.3. Elementary Discrete-Time Signals

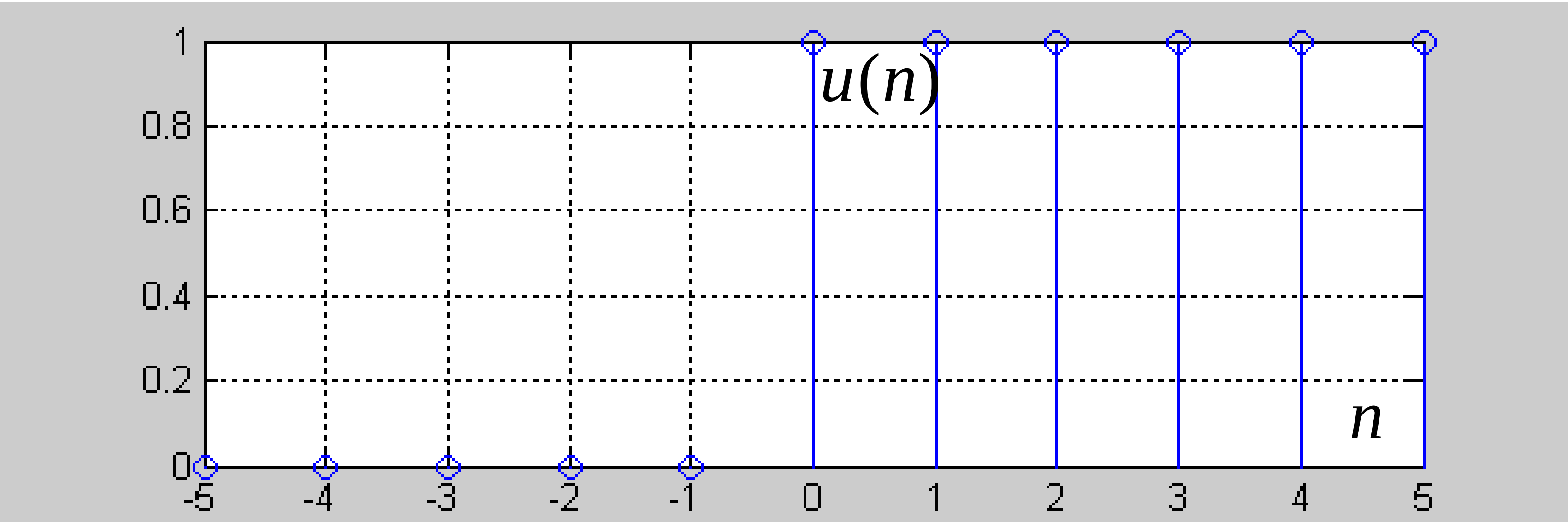
**A. Unit sample sequence** (unit sample, unit impulse, unit impulse signal)

$$\delta(n) = \begin{cases} 1 & for\ n = 0 \\ 0 & for\ n \neq 0 \end{cases}$$



**B. Unit step signal** (unit step, Heaviside step sequence)

$$u(n) = \begin{cases} 1 & for \quad n \geq 0 \\ 0 & for \quad n < 0 \end{cases}$$



## C. Complex-valued exponential signal

(complex sinusoidal sequence, complex phasor)

$$x(n) = e^{j\omega nT}, \ \left|x(n)\right| = 1, \ \arg\left[x(n)\right] = \omega nT = 2\pi f.nT = \frac{2\pi f.n}{f_S}$$

where

$$\omega \in R, \ n \in N, \ j = \sqrt{-1} \ \textit{is imaginary unit}$$

and

$T$ is sampling period and $f_S$ is sampling frequency.

## 1.1.2. Discrete-Time Systems. Definition

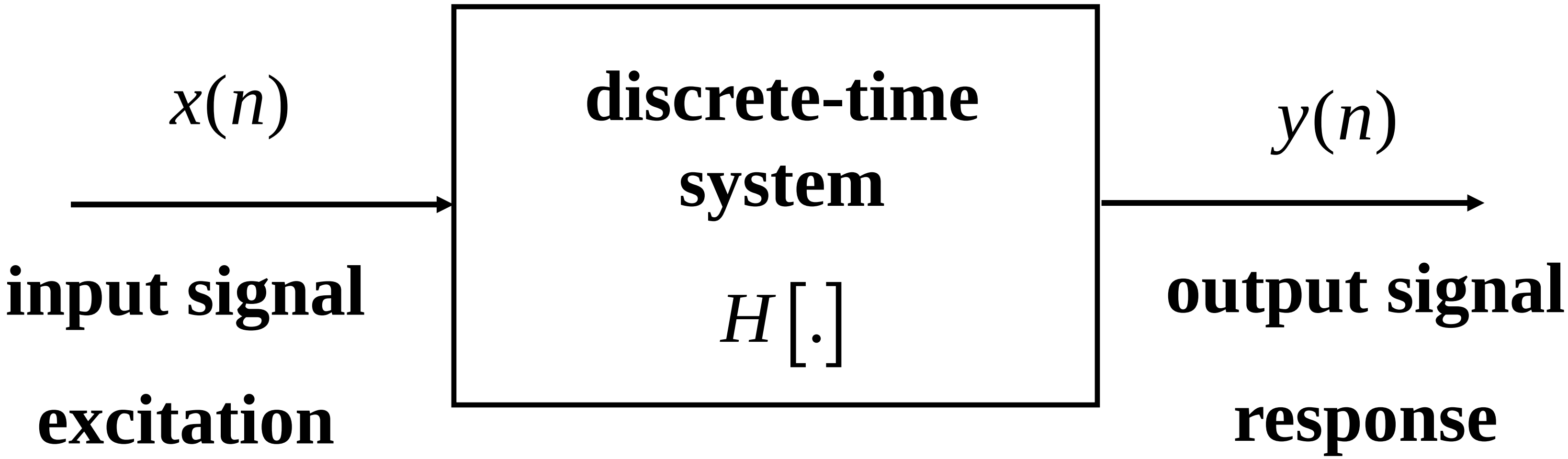
***A discrete-time system*** is a device or algorithm that operates on a discrete-time signal called ***the input*** or ***excitation*** (e.g. ***x(n)***), according ***to some rule*** (e.g. ***H[.]***) to produce another discrete-time signal called *the* ***output*** or ***response*** (e.g. ***y(n)***).

$$y(n) \equiv H\left[x(n)\right]$$

This expression denotes also **the transformation *H[.]* (also called operator or mapping) or processing performed by the system on *x(n)* to produce *y(n)*.**

# Input-Output Model of Discrete-Time System

(input-output relationship description)




$$y(n) \equiv H\left[x(n)\right]$$

$$x(n) \xrightarrow{H} y(n)$$

# 1.1.3. Classification of Discrete-Time Systems

## 1.1.3.1. Static vs. Dynamic Systems. Definition

A discrete-time system is called ***static*** or ***memoryless*** if its output at any time instant *n* depends on the input sample at the same time, but not on the past or future samples of the input. In the other case, the system is said to be ***dynamic*** or to have ***memory***.

If the output of a system at time *n* is completly determined by the input samples in the interval from *n-N* to *n* ( $N \geq 0$ ), the system is said to have memory of ***duration N***.

If $N = 0$, the system is ***static*** or ***memoryless***.

If $0 < N < \infty$, the system is said to have ***finite memory***.

If $N \to \infty$, the system is said to have ***infinite memory***.

# Examples:

The static (memoryless) systems:

$$y(n) = nx(n) + bx^3(n)$$

The dynamic systems with finite memory:

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k)$$

The dynamic system with infinite memory:

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

# 1.1.3.2. Time-Invariant vs. Time-Variable Systems. Definition

A discrete-time system is called ***time-invariant*** if its input-output characteristics do not change with time. In the other case, the system is called ***time-variable***.

**Definition.** A relaxed system $H[.]$ is ***time-*** or ***shift-invariant*** if only if

$$y(n) \equiv H\left[x(n)\right] \qquad x(n) \xrightarrow{H} y(n)$$

implies that

$$y(n-k) \equiv H\left[x(n-k)\right] \qquad x(n-k) \xrightarrow{H} y(n-k)$$

for ***every input signal*** $x(n)$ and ***every time shift*** $k$.

# Examples:

The time-invariant systems:

$$y(n) = x(n) + bx^3(n)$$

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k)$$

The time-variable systems:

$$y(n) = nx(n) + bx^3(n-1)$$

$$y(n) = \sum_{k=0}^{N} h^{N-n}(k)x(n-k)$$

## 1.1.3.3. Linear vs. Non-linear Systems. Definition

A discrete-time system is called ***linear*** if only if it satisfies the ***linear superposition principle***. In the other case, the system is called ***non-linear***.

**Definition.** A relaxed system $H[.]$ is ***linear*** if only if

$$H\left[a_1x_1(n)+a_2x_2(n)\right]=a_1H\left[x_1(n)\right]+a_2H\left[x_2(n)\right]$$

for any arbitrary input sequences $x_1(n)$ and $x_2(n)$, and any arbitrary constants $a_1$ and $a_2$.

# Examples:

The linear systems:

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k) \qquad y(n) = x(n^2) + bx(n-k)$$

The non-linear systems:

$$y(n) = nx(n) + bx^3(n-1) \qquad y(n) = \sum_{k=0}^{N} h(k)x(n-k)x(n-k+1)$$

## 1.1.3.4. Causal vs. Non-causal Systems. Definition

**Definition.** A system is said to be ***causal*** if the output of the system at any time $n$ (i.e., $y(n)$) depends only on present and past inputs (i.e., $x(n)$, $x(n-1)$, $x(n-2)$, ... ). In mathematical terms, the output of a ***causal*** system satisfies an equation of the form

$$y(n) = F\left[x(n), x(n-1), x(n-2), \cdots\right]$$

where $F[.]$ is some arbitrary function. If a system does not satisfy this definition, it is called ***non-causal***.

# Examples:

The causal system:

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k) \qquad y(n) = x^2(n) + bx(n-k)$$

The non-causal system:

$$y(n) = nx(n+1) + bx^3(n-1) \qquad y(n) = \sum_{k=-10}^{10} h(k)x(n-k)$$

## 1.1.3.5. Stable vs. Unstable of Systems. Definitions

An arbitrary relaxed system is said to be ***bounded input - bounded output (BIBO) stable*** if and only if every bounded input produces the bounded output. It means, that there exist some finite numbers say $M_x$ and $M_y$, such that

$$|x(n)| \leq M_x < \infty \Rightarrow |y(n)| \leq M_y < \infty$$

for all *n*. If for some bounded input sequence *x(n)* , the output *y(n)* is unbounded (infinite), the system is classified as ***unstable***.

# Examples:

The stable systems:

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k) \qquad y(n) = x(n^2) + 3x(n-k)$$

The unstable system:

$$y(n) = 3^n x^3(n-1)$$

# 1.1.3.6. Recursive vs. Non-recursive Systems. Definitions

A system whose output *y(n)* at time *n* depends on any number of the past outputs values ( e.g. *y(n-1), y(n-2),* ... ), is called a ***recursive system***. Then, the output of a causal recursive system can be expressed in general as

$$y(n) = F\left[y(n-1),\ y(n-2),\ \ldots,\ y(n-N),\ x(n),\ x(n-1),\ \ldots, x(n-M)\right]$$

where *F[.]* is some arbitrary function. In contrast, if *y(n)* at time *n* depends only on the present and past inputs

$$y(n) = F\left[x(n), x(n-1),\ \ldots, x(n-M)\right]$$

then such a system is called ***nonrecursive***.

## Examples:

The nonrecursive system:

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k)$$

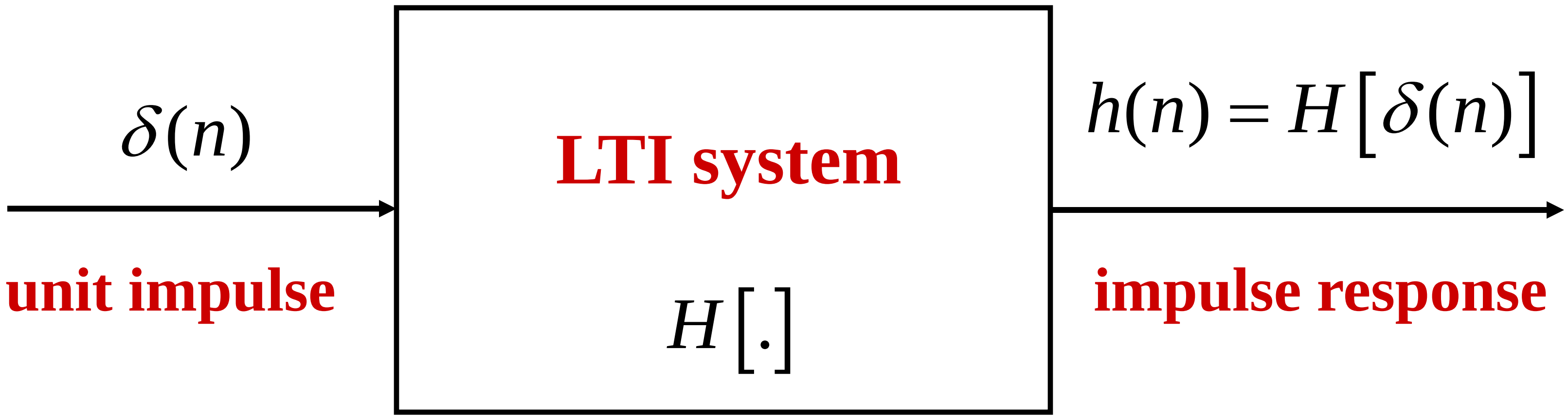
The recursive system:

$$y(n) = \sum_{k=0}^{N} b(k)x(n-k) - \sum_{k=1}^{N} a(k)y(n-k)$$

# 1.2. Linear-Discrete Time Time-Invariant Systems (LTI Systems)

## 1.2.1. Time-Domain Representation

# 1.2.1.1 Impulse Response and Convolution

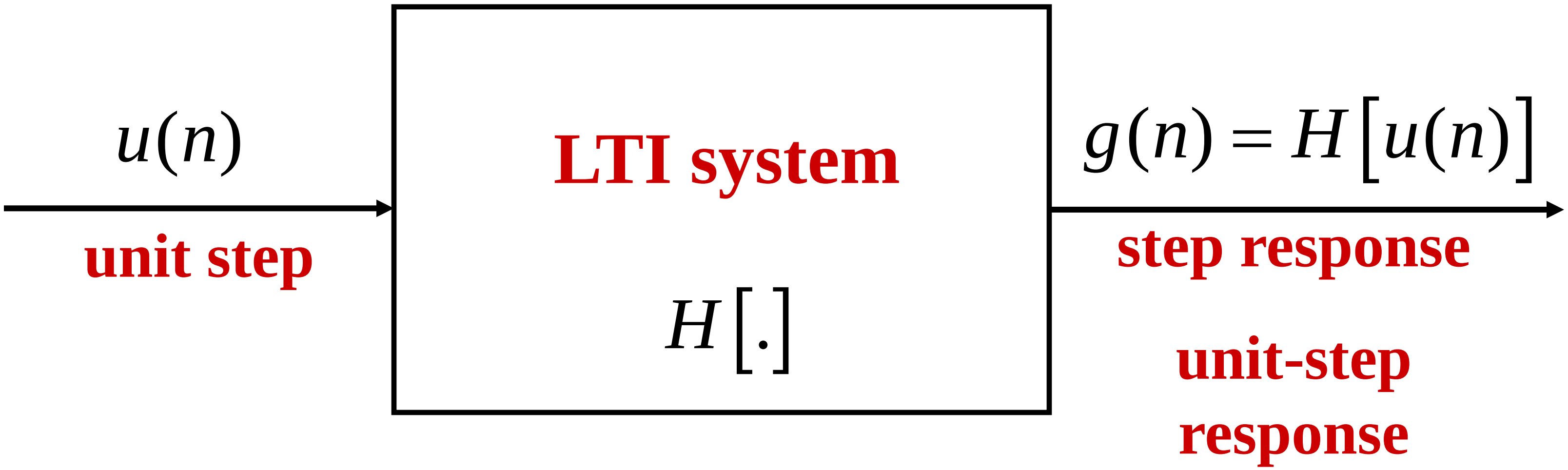



LTI system description by **convolution** (convolution sum):

$$y(n)=\sum_{k=-\infty}^{\infty}h(k)x(n-k)=\sum_{k=-\infty}^{\infty}x(k)h(n-k)=h(n)*x(n)=x(n)*h(n)$$

Viewed mathematically, the convolution operation satisfies the commutative law.

# 1.2.1.2. Step Response




$$g(n) = \sum_{k=-\infty}^{\infty} h(k)u(n-k) = \sum_{k=-\infty}^{n} h(k)$$

These expressions relate the impulse response to the step response of the system.

# 1.2.2. Impulse Response Property and Classification of LTI Systems

## 1.2.2.1. Causal LTI Systems

A relaxed LTI system is ***causal*** if and only if its impulse response is zero for negative values of $n$ , i.e.

$$h(n) = 0 \quad for \quad n < 0$$

Then, the two equivalent forms of the convolution formula can be obtained for the causal LTI system:

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) = \sum_{k=-\infty}^{n} x(k)h(n-k)$$

## 1.2.2.2. Stable LTI Systems

A LTI system is ***stable*** if its impulse response is absolutely summable, i.e.

$$\sum_{k=-\infty}^{\infty} |h(k)|^2 < \infty$$

# 1.2.2.3. Finite Impulse Response (FIR) LTI Systems and Infinite Impulse Response (IIR) LTI Systems

Causal **FIR** LTI systems: $$y(n)=\sum_{k=0}^{N}h(k)x(n-k)$$

**IIR** LTI systems: $$y(n)=\sum_{k=0}^{\infty}h(k)x(n-k)$$

# 1.2.2.4. Recursive and Nonrecursive LTI Systems

Causal ***nonrecursive*** LTI:

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k)$$

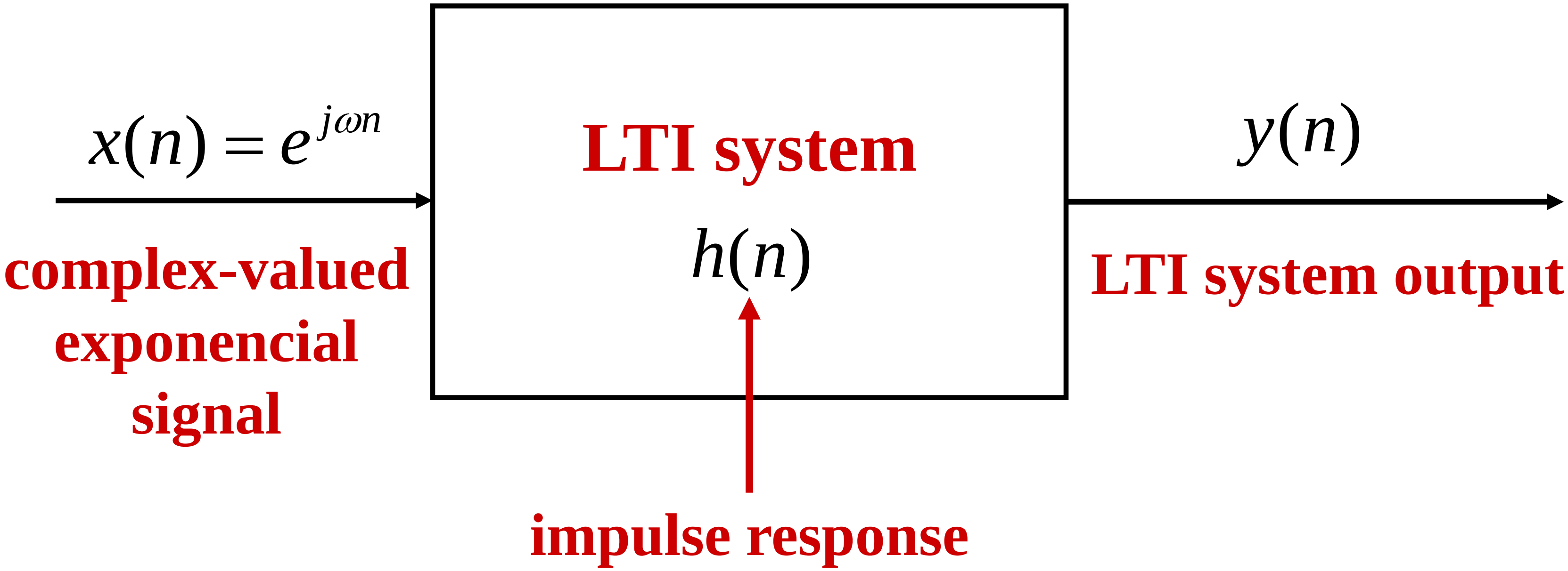
Causal ***recursive*** LTI:

$$y(n) = \sum_{k=0}^{N} b(k)x(n-k) - \sum_{k=1}^{M} a(k)y(n-k)$$

LTI systems:

characterized by ***constant-coefficient difference equations***

# 1.3. Frequency-Domain Representation of Discrete Signals and LTI Systems




$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

LTI system output:

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = \sum_{k=-\infty}^{\infty} h(k)e^{j\omega(n-k)} =$$

$$= \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}e^{j\omega n} = e^{j\omega n}\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$$

$$y(n) = e^{j\omega n}H(e^{j\omega})$$

***Frequency response:*** $H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$

$$H(e^{j\omega}) = \left|H(e^{j\omega})\right| e^{j\phi(\omega)}$$

$$H(e^{j\omega}) = \mathrm{Re}\left[H(e^{j\omega})\right] + j\,\mathrm{Im}\left[H(e^{j\omega})\right]$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)\cos\omega k + j\left[-\sum_{k=-\infty}^{\infty} h(k)\sin\omega k\right]$$

$$\mathrm{Re}\left[H(e^{j\omega})\right] = \sum_{k=-\infty}^{\infty} h(k)\cos\omega k$$

$$\mathrm{Im}\left[H(e^{j\omega})\right] = -\sum_{k=-\infty}^{\infty} h(k)\sin\omega k$$

***Magnitude response:***

$$\left|H(e^{j\omega})\right| = \sqrt{\mathrm{Re}\left[H(e^{j\omega})\right]^2 + \mathrm{Im}\left[H(e^{j\omega})\right]^2}$$

***Phase response:***

$$\phi(\omega) = \arg\left[H(e^{j\omega})\right] = arctg\,\frac{\mathrm{Im}\left[H(e^{j\omega})\right]}{\mathrm{Re}\left[H(e^{j\omega})\right]}$$

***Group delay function:***

$$\tau(\omega) = -\frac{d\phi(\omega)}{d\omega}$$

## 1.3.1. Comments on relationship between the impulse response and frequency response

The important property of ***the frequency response***

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} = \sum_{k=-\infty}^{\infty} h(k)e^{-j[\omega+2l\pi]} = H(e^{j[\omega+2l\pi]})$$

is fact that this function ***is periodic with period*** $2\pi$.

In fact, we may view the previous expression as the exponential Fourier series expansion for $H(e^{j\omega})$, with *h(k)* as the Fourier series coefficients. Consequently, the unit impulse response *h(k)* is related to $H(e^{j\omega})$ through the integral expression

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega$$

**1.3.2. Comments on symmetry properties**

For LTI systems with real-valued impulse response, the magnitude response, phase responses, the real component of and the imaginary component of $H(e^{j\omega})$ possess these symmetry properties:

**The real component**: ***even function*** of $\omega$ periodic with period $2\pi$

$$\mathrm{Re}\left[H(e^{-j\omega})\right] = \mathrm{Re}\left[H(e^{j\omega})\right]$$

**The imaginary component**: ***odd function*** of $\omega$ periodic with period $2\pi$

$$\mathrm{Im}\left[H(e^{-j\omega})\right] = -\mathrm{Im}\left[H(e^{j\omega})\right]$$

**The magnitude response:** even function of $\omega$ periodic with period $2\pi$

$$\left|H(e^{j\omega})\right| = \left|H(e^{-j\omega})\right|$$

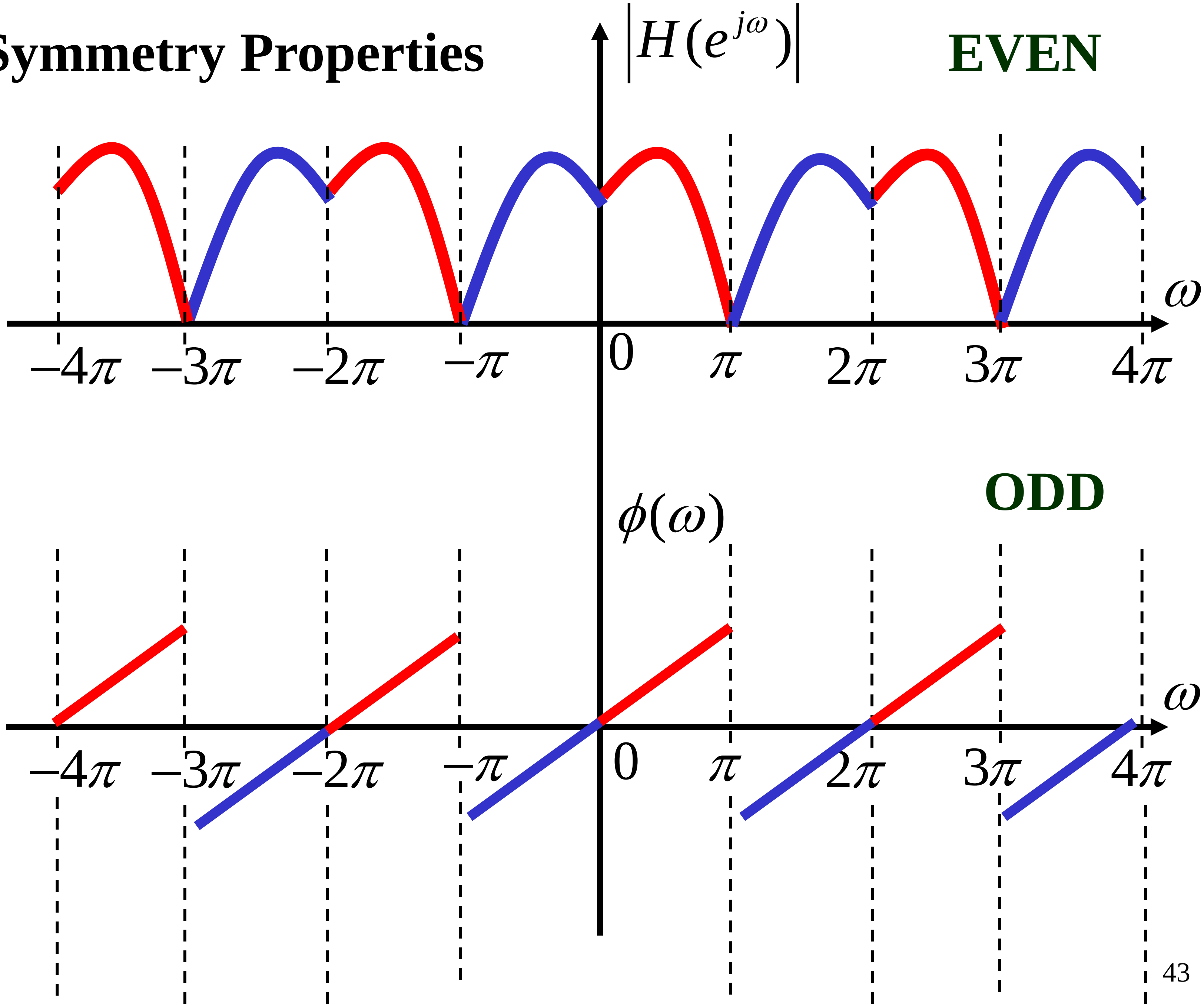
**The phase response:** odd function of $\omega$ periodic with period $2\pi$

$$\arg\left[H(e^{-j\omega})\right] = -\arg\left[H(e^{j\omega})\right]$$

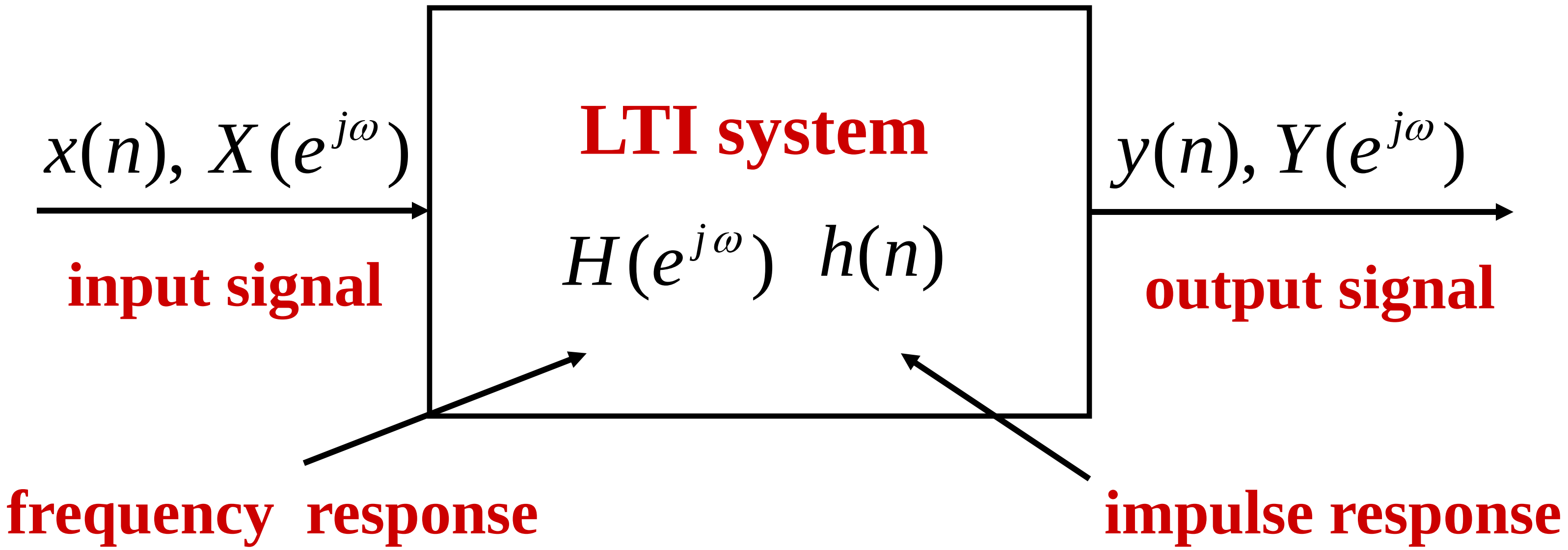
**Consequence:**

If we known $\left|H(e^{j\omega})\right|$ and $\phi(\omega)$ for $0 \le \omega \le \pi,$ we can describe these functions ( i.e. also $H(e^{j\omega})$ ) for all values of $\omega$ .

# Symmetry Properties



## 1.3.3. Comments on Fourier Transform of Discrete Signals and Frequency-Domain Description of LTI Systems



**The input signal *x(n)* and the spectrum of *x(n)*:**

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x(k)e^{-j\omega k} \qquad x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

**The output signal *y(n)* and the spectrum of *y(n)*:**

$$Y(e^{j\omega}) = \sum_{k=-\infty}^{\infty} y(k)e^{-j\omega k} \qquad y(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})e^{j\omega n}d\omega$$

**The impulse response *h(n)* and the spectrum of *h(n)*:**

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} \qquad h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega$$

**Frequency-domain description of LTI system:**

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

## 1.3.4. Comments on Normalized Frequency

It is often desirable to express the frequency response of an LTI system in terms of units of frequency that involve sampling interval *T*. In this case, the expressions:

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} \qquad h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega$$

are modified to the form:

$$H(e^{j\omega T}) = \sum_{k=-\infty}^{\infty} h(kT)e^{-j\omega kT}$$

$$h(nT) = \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} H(e^{j\omega T})e^{j\omega nT}d\omega$$

$H(e^{j\omega T})$ is periodic with period $2\pi/T = 2\pi F$, where $F$ is sampling frequency.

Solution: **normalized frequency approach**: $F/2 \rightarrow \pi$

**Example:**

$$F = 100kHz \quad F/2 = 50kHz \quad 50kHz \rightarrow \pi$$

$$f_1 = 20kHz \qquad \omega_1 = \frac{20x10^3}{50x10^3}\pi = \frac{2\pi}{5} = 0.4\pi$$

$$f_2 = 25kHz \qquad \omega_2 = \frac{25x10^3}{50x10^3}\pi = \frac{\pi}{2} = 0.5\pi$$

# 1.4. Transform-Domain Representation of Discrete Signals and LTI Systems

## 1.4.1. Z -Transform

**Definition:** The $Z$ – transform of a discrete-time signal $x(n)$ is defined as the power series:

$$X(z) = \sum_{k=-\infty}^{\infty} x(n) z^{-k} \qquad X(z) = Z[x(n)]$$

where $z$ is a complex variable. The above given relations are sometimes called **the direct *Z* - transform** because they transform the time-domain signal $x(n)$ into its complex-plane representation $X(z)$.

Since $Z$ – transform is an infinite power series, it exists only for those values of $z$ for which this series converges. The **region of convergence** of $X(z)$ is the set of all values of $z$ for which $X(z)$ attains a finite value.

The procedure for transforming from z – domain to the time-domain is called **the inverse Z – transform**. It can be shown that the inverse *Z* – transform is given by

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \qquad x(n) = Z^{-1}\left[X(z)\right]$$

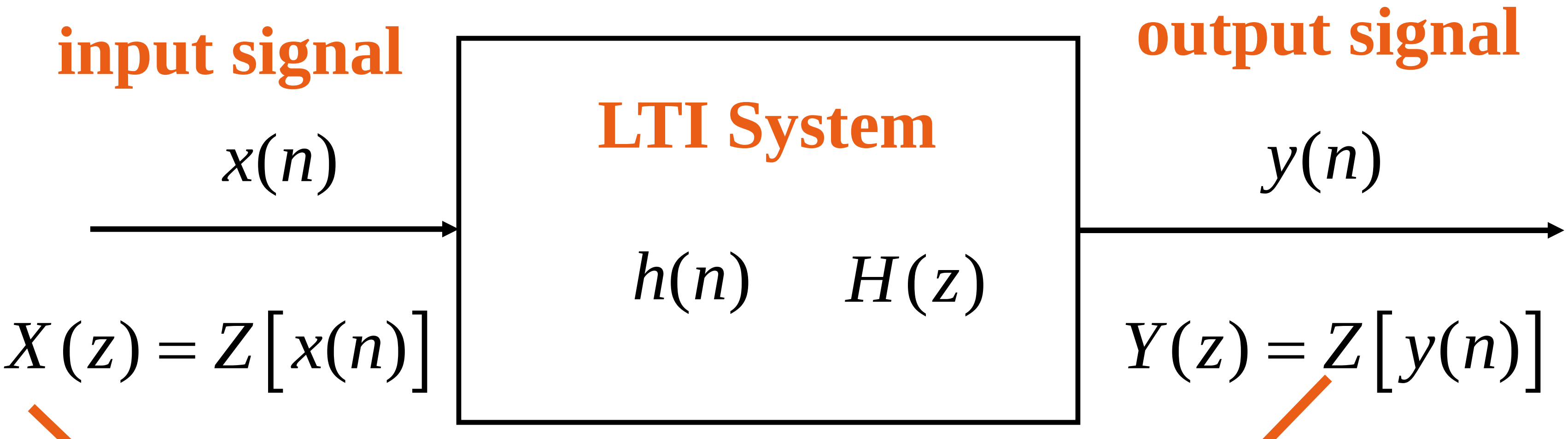
where *C* denotes the closed contour in the region of convergence of *X(z)* that encircles the origin.

## 1.4.2. Transfer Function

The LTI system can be described by means of **a constant coefficient linear difference equation** as follows

$$y(n) = \sum_{k=0}^{N} b(k)x(n-k) - \sum_{k=1}^{M} a(k)y(n-k)$$

Application of the $Z$-transform to this equation under zero initial conditions leads to the notion of **a transfer function**.




$$H(z)=Z[h(n)]$$

**Transfer function:** the ratio of the $Z$ - transform of the output signal and the $Z$ - transform of the input signal of the LTI system:

$$H(z)=\frac{Y(z)}{X(z)}=\frac{Z[y(n)]}{Z[x(n)]}$$

LTI system: the *Z*-transform of the constant coefficient linear difference equation under zero initial conditions:

$$y(n)=\sum_{k=0}^{N}b(k)x(n-k)-\sum_{k=1}^{M}a(k)y(n-k)$$

$$Y(z)=\sum_{k=0}^{N}b(k)z^{-k}X(z)-\sum_{k=1}^{M}a(k)z^{-k}Y(z)$$

The transfer function of the LTI system:

$$H(z)=\frac{Y(z)}{X(z)}=\frac{\sum_{k=0}^{N}b(k)z^{-k}}{1+\sum_{k=1}^{M}a(k)z^{-k}}$$

*H(z)*: may be viewed as a rational function of a complex variable *z* ($z^{-1}$).

# 1.4.3. Poles, Zeros, Pole-Zero Plot

Let us assume that *H(z)* has been expressed in its irreducible or so-called factorized form:

$$H(z)=\frac{\sum_{k=0}^{N} b(k)z^{-k}}{1+\sum_{k=1}^{M} a(k)z^{-k}}=\frac{b_0}{a_0}z^{N-M}\frac{\prod_{k=1}^{N}(z-z_k)}{\prod_{k=1}^{M}(z-p_k)}$$

**Zeros of *H(z)*:** the set $\{z_k\}$ of *z*-plane for which $H(z_k)=0$

**Poles of *H(z)*:** the set $\{p_k\}$ of $z$ -plane for which $H(p_k)\rightarrow\infty$

**Pole-zero plot:** the plot of **the zeros** and **the poles** of *H(z)* in the z-plane represents a strong tool for LTI system description.

**Example:** the 4-th order Butterworth low-pass filter, cut off frequency $\omega_1 = \pi/3$.

$b$ =[ 0.0186 0.0743 0.1114 0.0743 0.0186 ]

$a$ =[ 1.0000 -1.5704 1.2756 -0.4844 0.0762 ]

$$H(z) = \frac{\sum_{k=0}^{N} b(k) z^{-k}}{1 + \sum_{k=1}^{M} a(k) z^{-k}}$$

$$H(z) = \frac{\sum_{k=0}^{N} b(k) z^{-k}}{1 + \sum_{k=1}^{M} a(k) z^{-k}}$$

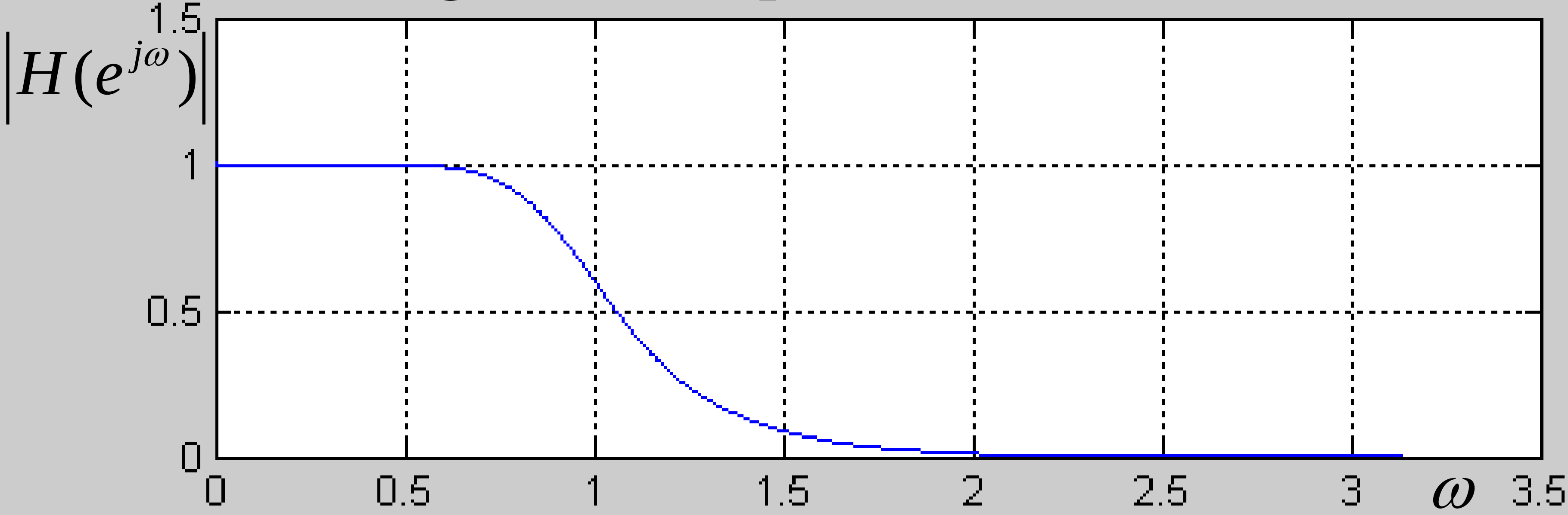
$z_1$= -1.0002, $z_2$= -1.0000 + 0.0002$j$

$z_3$= -1.0000 - 0.0002$j$, $z_4$= -0.9998

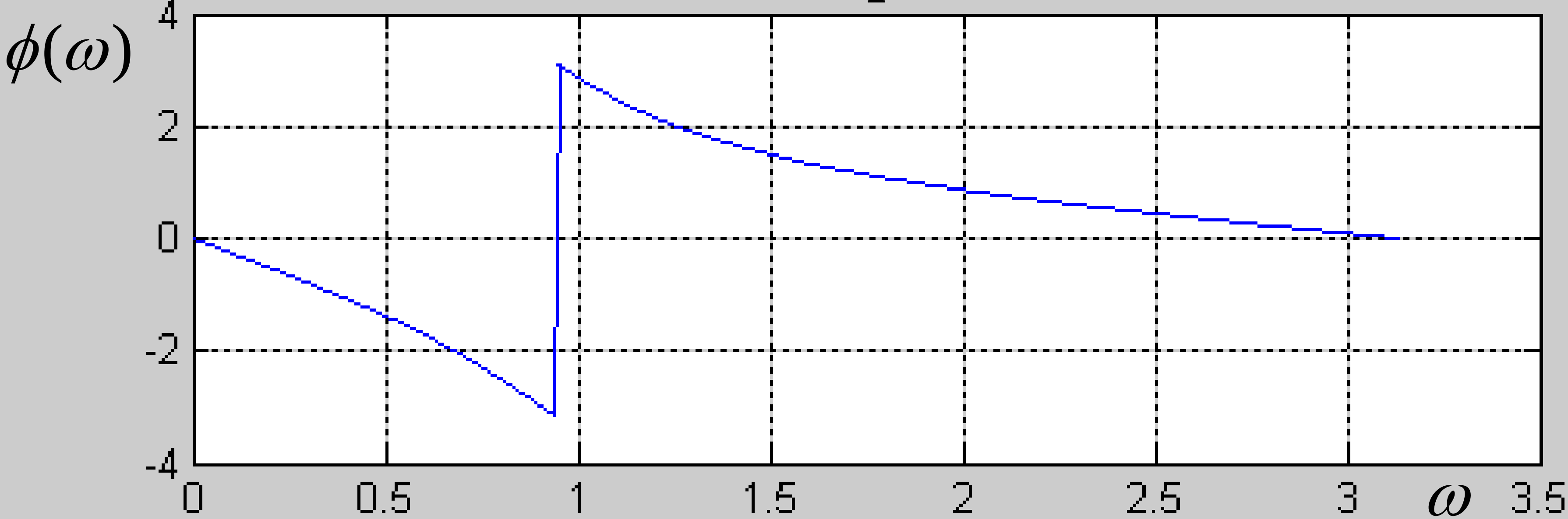
$p_1$= 0.4488 + 0.5707$j$, $p_2$= 0.4488 - 0.5707$j$

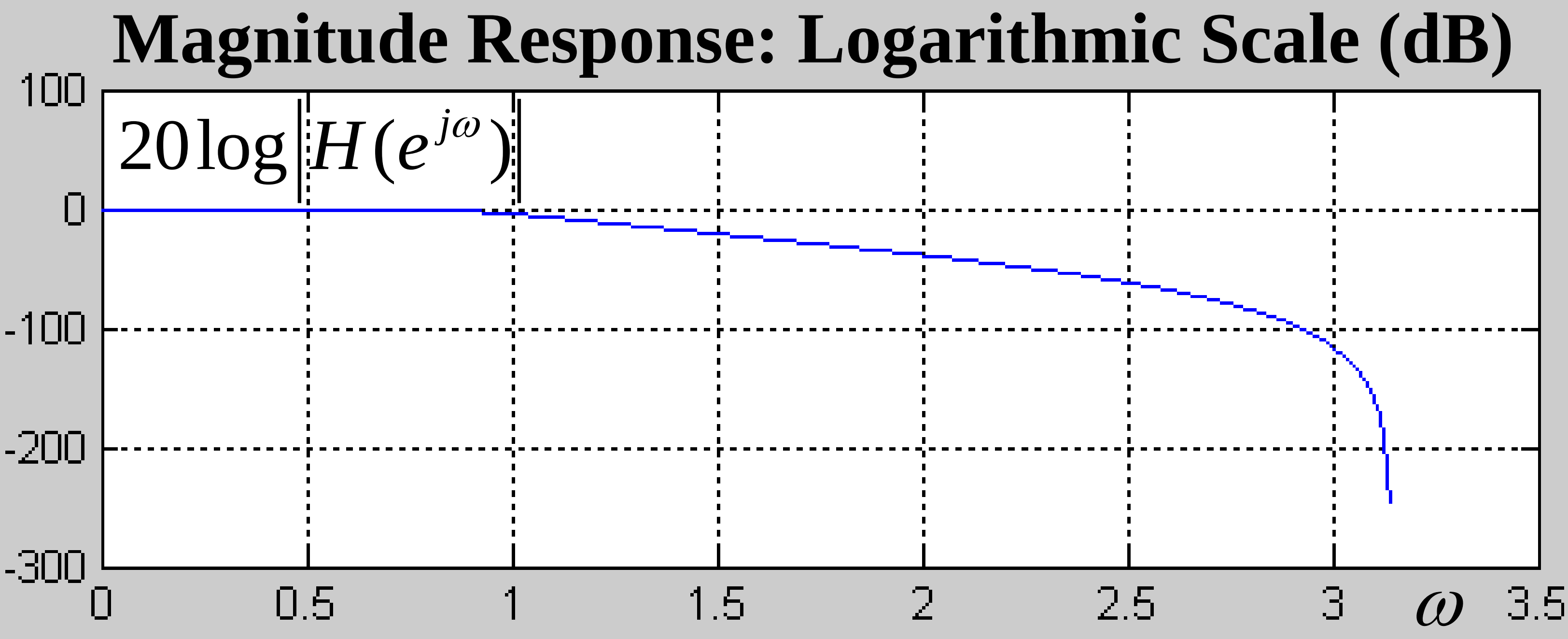
$p_3$= 0.3364 + 0.1772$j$, $p_4$= 0.3364 - 0.1772$j$

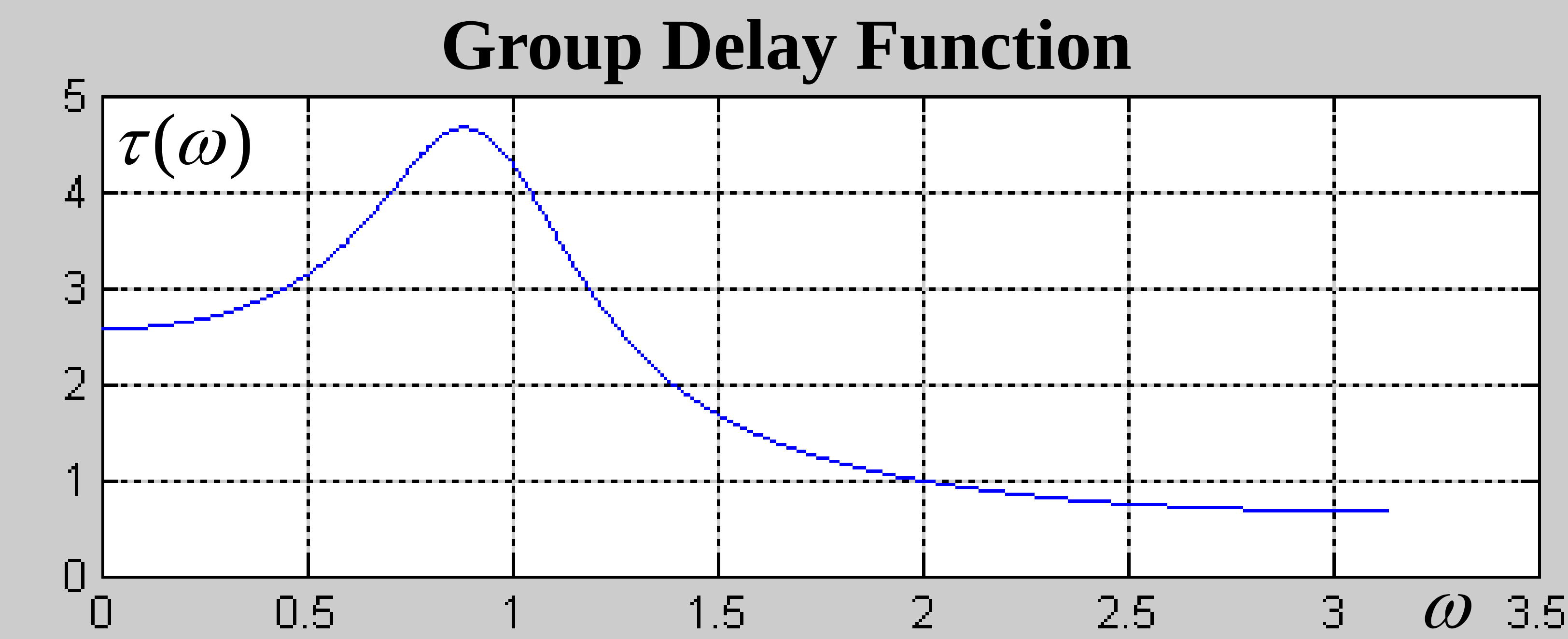
# Magnitude Response: Linear Scale



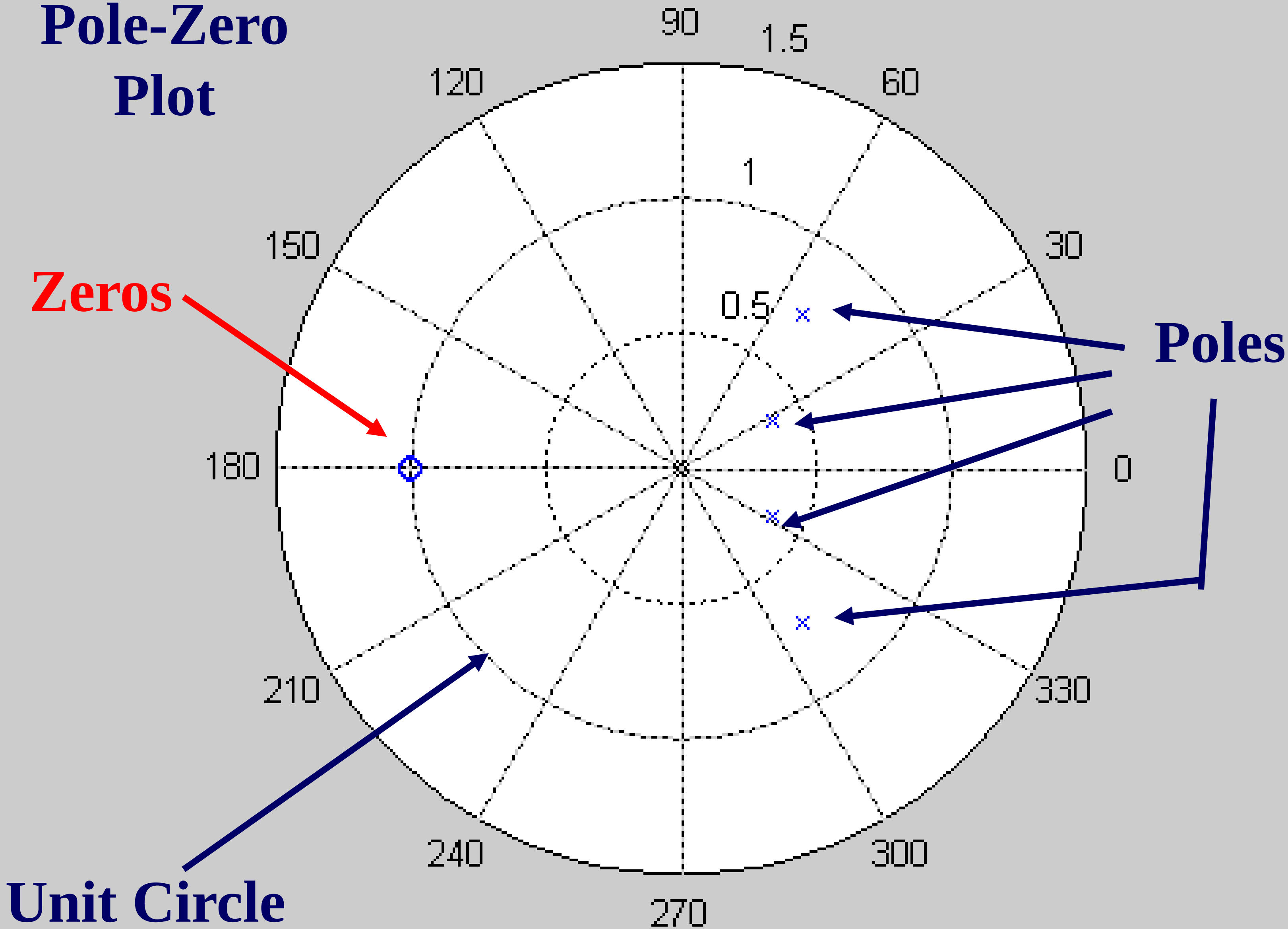

# Phase Response



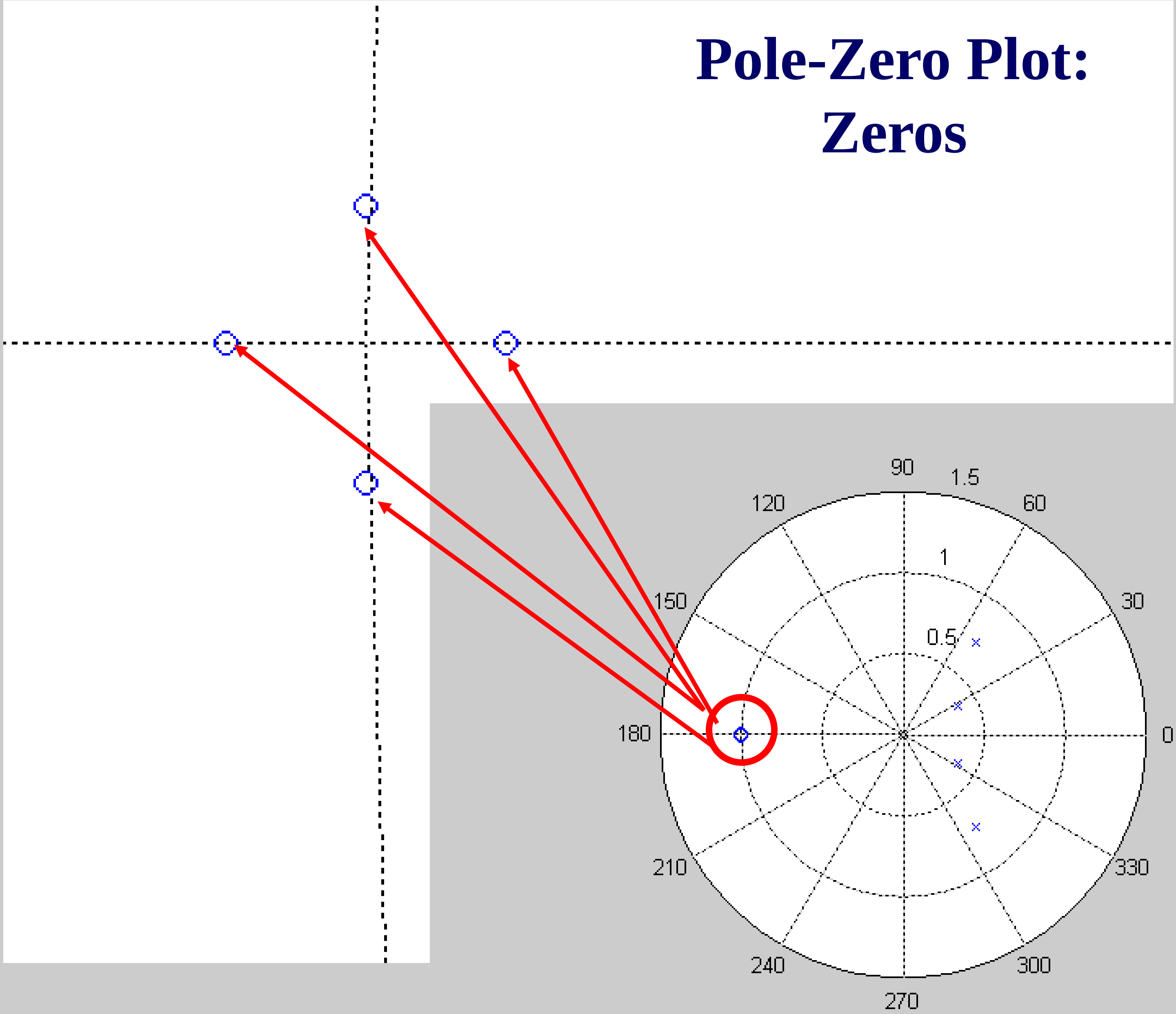

Magnitude Response: Logarithmic Scale (dB)
$20\log|H(e^{j\omega})|$
100
0
-100
-200
-300
0
0.5
1
1.5
2
2.5
3
$\omega$
3.5


Group Delay Function
$\tau(\omega)$
5
4
3
2
1
0
0
0.5
1
1.5
2
2.5
3
$\omega$
3.5

# Pole-Zero Plot



# Pole-Zero Plot: Zeros



## 1.4.4. Transfer Function and Stability of LTI Systems

**Condition: LTI system is BIBO stable if and only if** the unit circle falls within the region of convergence of the power series expansion for its transfer function. In the case when the transfer function characterizes a causal LTI system, the stability condition is equivalent to the requirement that **the transfer function *H(z)* has all of its poles inside the unit circle.**

## Example 1: stable system

$$H(z)=\frac{1-0.9z^{-1}+0.18z^{-2}}{1-0.8z^{-1}+0.64z^{-2}}$$

$z_1=0.3 \quad p_1=0.4000+0.6928j \quad |p_1|=0.8<1$

$z_2=0.6 \quad p_2=0.4000-0.6928j \quad |p_2|=0.8<1$

## Example 2: unstable system

$$H(z)=\frac{1-0.16z^{-2}}{1-1.1z^{-1}+1.21z^{-2}}$$

$z_1=0.4 \quad p_1=0.5500+0.9526j \quad |p_1|=1.1>1$

$z_2=-0.4 \quad p_2=0.5500-0.9526j \quad |p_2|=1.1>1$

## 1.4.5. LTI System Description. Summary

**Time – Domain:**

constant coefficient linear difference equation

$$y(n) = \sum_{k=0}^{N} b(k)x(n-k) - \sum_{k=1}^{M} a(k)y(n-k)$$

$h(n)$

**Z – Domain:**

transfer function

Z

$Z^{-1}$

$$H(z) = \frac{\sum_{k=0}^{N} b(k)z^{-k}}{1+\sum_{k=1}^{M} a(k)z^{-k}}$$

**Frequency – Domain:**

FT

frequency response

$FT^{-1}$

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{N} b(k)e^{-j\omega k}}{1+\sum_{k=1}^{M} a(k)e^{-j\omega k}}$$

$z = e^{j\omega}$ $\quad e^{j\omega} = z$

**Time – Domain:** impulse response $h(k)$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k} \qquad H(z) = \sum_{k=-\infty}^{\infty} h(k) z^{-k}$$

**Z – Domain:** transfer function $H(z)$

$$H\left(e^{j\omega}\right) = H\left(z\right)_{z=e^{jw}} \qquad h(n) = \frac{1}{2\pi j} \oint_C H(z) z^{n-1} dz$$

**Frequency – Domain:** frequency response $H\left(e^{j\omega}\right)$

$$H(z) = H\left(e^{j\omega}\right)_{e^{j\omega}=z} \qquad h(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega k} d\omega$$